

# Strongly residual coordinates over $A[x]$

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## Abstract

For a domain  $A$  of characteristic zero, a polynomial  $f \in A[x]^{[n]}$  is called a *strongly residual coordinate* if  $f$  becomes a coordinate (over  $A$ ) upon going modulo  $x$ , and  $f$  becomes a coordinate (over  $A[x, x^{-1}]$ ) upon inverting  $x$ . We study the question of when a strongly residual coordinate in  $A[x]^{[n]}$  is a coordinate, a question closely related to the Dolgachev-Weisfeiler conjecture. It is known that all strongly residual coordinates are coordinates for  $n = 2$ . We show that a large class of strongly residual coordinates that are generated by elementaries over  $A[x, x^{-1}]$  are in fact coordinates for arbitrary  $n$ , with a stronger result in the  $n = 3$  case. As an application, we show that all Vénéreau-type polynomials are 1-stable coordinates.

## 1 Introduction

Let  $A$  (and all other rings) be a commutative ring with one. An  $A$ -coordinate (if  $A$  is understood, we simply say *coordinate*; some authors prefer the term *variable*) is a polynomial  $f \in A^{[n]}$  for which there exist  $f_2, \dots, f_n \in A^{[n]}$  such that  $A[f, f_2, \dots, f_n] = A^{[n]}$ . It is natural to ask when a polynomial is a coordinate; this question is extremely deep and has been studied for some time. There are several longstanding conjectures giving a criteria for a polynomial to be a coordinate:

**Conjecture 1** (Abhyankar-Sathaye). *Let  $A$  be a ring of characteristic zero, and let  $f \in A^{[n]}$ . If  $A^{[n]}/(f) \cong A^{[n-1]}$ , then  $f$  is an  $A$ -coordinate.*

**Conjecture 2** (Dolgachev-Weisfeiler). *Suppose  $A = \mathbb{C}^{[r]}$ , and let  $f \in A^{[n]}$ . If  $A[f] \hookrightarrow A^{[n]}$  is an affine fibration, then  $f$  is an  $A$ -coordinate.*

**Conjecture 3.** *Let  $A$  be a ring of characteristic zero, and let  $f \in A^{[n]}$ . If  $f$  is a coordinate in  $A^{[n+m]}$  for some  $m > 0$ , then  $f$  is a coordinate in  $A^{[n]}$ .*

The Abhyankar-Sathaye conjecture is known only for  $A$  a field and  $n = 2$  (due to Abhyankar and Moh [1] and Suzuki [15], independently). The  $n = 2$  case of the Dolgachev-Weisfeiler conjecture follows from results of Asanuma [2] and Hamann [7]. The case where both  $n = 3$  and  $A = \mathbb{C}$  follows from a theorem of Sathaye [13]; see [5] for more details on the background of the Dolgachev-Weisfeiler conjecture.

There are several examples of polynomials satisfying the hypotheses of these conjectures whose status as a coordinate remains open. Many are constructed via a slight variation of the following classical method for constructing exotic automorphisms of  $A^{[n]}$ : let  $x \in A$  be a nonzero divisor. One may easily construct elementary automorphisms (those that fix  $n - 1$  variables) of  $A_x^{[n]}$ ; then, one can carefully compose these automorphisms (over  $A_x$ ) to produce an endomorphism of  $A^{[n]}$ . It is a simple application of the formal inverse function theorem to see that such maps must, in fact, be automorphisms of  $A^{[n]}$ . The well known Nagata map arises in this manner:

$$\begin{aligned}\sigma &= (y + x(xz - y^2), z + 2y(xz - y^2) + x(xz - y^2)^2) \\ &= (y, z + \frac{y^2}{x}) \circ (y + x^2z, z) \circ (y, z - \frac{y^2}{x})\end{aligned}$$

While the Nagata map is generated over  $\mathbb{C}[x, x^{-1}]$  by three elementary automorphisms, Shestakov and Umirbaev [14] famously proved that it is wild (i.e. not generated by elementary and linear automorphisms) as an automorphism of  $\mathbb{C}[x, y, z]$  over  $\mathbb{C}$ .

When interested in producing exotic polynomials, we may relax the construction somewhat; let  $y$  be a variable of  $A^{[n]}$ , and compose elementary automorphisms of  $A_x^{[n]}$  until the resulting map has its  $y$ -component in  $A^{[n]}$ . For example, the Vénéreau polynomial  $f = y + x(xz + y(yu + z^2))$  arises as the  $y$ -component of the following automorphism over  $\mathbb{C}[x, x^{-1}]$

$$\phi = (y + x^2z, z, u) \circ (y, z + \frac{y(yu + z^2)}{x}, u - \frac{2z(yu + z^2)}{x} - y(yu + z^2)^2) \quad (1)$$

This type of construction motivates the following definition:

**Definition 1.** A polynomial  $f \in A[x]^{[n]}$  is called a *strongly residual coordinate* if  $f$  is a coordinate over  $A[x, x^{-1}]$  and if  $\bar{f}$ , the image modulo  $x$ , is a coordinate over  $A$ .

The Vénéreau polynomial is perhaps the most widely known example of a strongly residual coordinate that satisfies the hypotheses of the three conjectures (with  $A = \mathbb{C}[x]$ ), yet it is an open question whether it is a coordinate (see [17], [8], [6], and [11], among others, for more on that particular question).

One may observe that the second automorphism in the above composition (1) is essentially the Nagata map, and is wild over  $\mathbb{C}[x, x^{-1}]$ . The wildness of this map is a crucial difficulty in resolving the status of the Vénéreau polynomial. Our present goal is to show that a large class of strongly residual coordinates generated by maps that are elementary over  $\mathbb{C}[x, x^{-1}]$  are coordinates. Our methods are quite constructive and algorithmic, although the computations can become unwieldy quite quickly. One application is to show that all Vénéreau-type polynomials, a generalization of the Vénéreau polynomial studied by the author in [11], are one-stable coordinates (coming from the fact that the Nagata map is one-stably tame). Additionally, we also very quickly recover a result of Russell (Corollary 6) on coordinates in 3 variables over a field of characteristic zero.

## 2 Preliminaries

Throughout, we set  $R = A[x]$  and  $S = R_x = A[x, x^{-1}]$ . We adopt the standard notation for automorphism groups of the polynomial ring  $A^{[n]} = A[z_1, \dots, z_n]$ :

1.  $\text{GA}_n(A)$  denotes the general automorphism group  $\text{Aut}_{\text{Spec } A}(\text{Spec } A^{[n]})$ , which is antiisomorphic to  $\text{Aut}_A A^{[n]}$  (some authors choose to define it as the latter).
2.  $\text{EA}_n(A)$  denotes the subgroup generated by the elementary automorphisms; that is, those fixing  $n - 1$  variables.
3.  $\text{TA}_n(A) = \langle \text{EA}_n(A), \text{GL}_n(A) \rangle$  is the tame subgroup.
4.  $\text{D}_n(A) \leq \text{GL}_n(A)$  is the subgroup of diagonal matrices.
5.  $\text{P}_n(A) \leq \text{GL}_n(A)$  is the subgroup of permutation matrices.
6.  $\text{GP}_n(A) = \text{D}_n(A)\text{P}_n(A) \leq \text{GL}_n(A)$  is the subgroup of generalized permutation matrices.

We also make one non-standard definition when working over  $R = A[x]$ :

7.  $\text{IA}_n(R) = \{\phi \in \text{GA}_n(R) \mid \phi \equiv \text{id} \pmod{x}\}$  is the subgroup of all automorphisms that are equal to the identity modulo  $x$ . It is the kernel of the natural map  $\text{GA}_n(R) \rightarrow \text{GA}_n(A)$ .

*Remark 1.* In fact, the surjection  $\text{GA}_n(R) \rightarrow \text{GA}_n(A)$  splits (by the natural inclusion), so we have  $\text{GA}_n(R) \cong \text{IA}_n(R) \rtimes \text{GA}_n(A)$ .

**Definition 2.** Let  $f_1, \dots, f_m \in R^{[n]}$ .

1.  $(f_1, \dots, f_m)$  is called a *partial system of coordinates* (over  $R$ ) if there exists  $g_{m+1}, \dots, g_n \in R^{[n]}$  such that  $(f_1, \dots, f_m, g_{m+1}, \dots, g_n) \in \text{GA}_n(R)$ .
2.  $(f_1, \dots, f_m)$  is called a *partial system of residual coordinates* if  $R[f_1, \dots, f_m] \hookrightarrow R^{[n]}$  is an affine fibration; that is,  $R^{[n]}$  is flat over  $R[f_1, \dots, f_m]$  and for each prime ideal  $\mathfrak{p} \in \text{Spec } R[f_1, \dots, f_m]$ ,  $R^{[n]} \otimes_{R[f_1, \dots, f_m]} \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^{[n-m]}$ .
3.  $(f_1, \dots, f_m)$  is called a *partial system of strongly  $x$ -residual coordinates* if  $(f_1, \dots, f_m)$  is a partial system of coordinates over  $S$  and  $(\bar{f}_1, \dots, \bar{f}_m)$ , the images modulo  $x$ , is a partial system of coordinates over  $A = \bar{R} = R/xR$ . If  $x$  is understood, we may simply say *strongly residual coordinate*.

A single polynomial is called a *coordinate* (respectively *residual coordinate*, *strongly residual coordinate*) when  $m = 1$  in the above definitions.

*Remark 2.* If  $A$  is a field, then strongly residual coordinates are residual coordinates.

In light of this definition, the Dolgachev-Weisfeiler conjecture can be stated in this context as

**Conjecture 4.** *Partial systems of residual coordinates are partial systems of coordinates*

Similarly, we have

**Conjecture 5.** *Partial systems of strongly residual coordinates are partial systems of coordinates.*

Our main focus will be on constructing and identifying strongly residual coordinates that are coordinates, although in some cases our methods will generalize slightly to partial systems of coordinates. While we lose some generality as compared to considering residual coordinates, we are able to use some very constructive approaches. We first give a short, direct proof of the  $n = 2$  case (for coordinates) that shows the flavor of our methods:

**Theorem 1.** *Let  $A$  be an integral domain of characteristic zero, and  $R = A[x]$ . Let  $f \in R^{[2]}$  be a strongly residual coordinate. Then  $f$  is a coordinate.*

*Proof.* Since  $\bar{f}$  is a coordinate in  $\bar{R}^{[2]} = \bar{R}[y, z]$ , without loss of generality we may assume  $f = y + xQ$  for some  $Q \in R[y, z]$ . Since  $f$  is an  $S$ -coordinate, perhaps after composing with a linear map, we obtain some  $\phi = (y + xQ, z + x^{-t}P) \in \text{GA}_2(S)$  with  $J\phi = 1$  and  $P \in R^{[2]} \setminus xR^{[2]}$ . We inductively show that such a map  $\phi$  is elementarily (over  $S$ ) equivalent to a map with  $t \leq 0$ , which gives an element of  $\text{GA}_2(R)$ . We compute

$$J\phi = J(y, z) + xJ(Q, z) + x^{1-t}J(Q, P) + x^{-t}J(y, P)$$

Since  $J\phi = 1$ , we have  $xJ(Q, z) + x^{1-t}J(Q, P) + x^{-t}J(y, P) = 0$ . Thus, comparing  $x$ -degrees, we must have  $J(y, P) \in xR^{[2]}$ . This means  $P = P_0(y) + xP_1$  for some  $P_1 \in R^{[2]}$ . Then we have  $(y, z - x^{-t}P_0(y)) \circ \phi = (y + xQ, z + x^{-t+1}P')$  for some  $P' \in R^{[2]}$  by Taylor's formula, allowing us to apply the inductive hypothesis.  $\square$

*Remark 3.* Analogous results for residual coordinates are due to Kambayashi and Miyanishi [9] and Kambayashi and Wright [10].

The  $n = 3$  case remains open, with the Vénéreau polynomial providing the most widely known example of a strongly residual coordinate whose status as a coordinate has not been determined.

We next describe some notation necessary to state the most general form of our results.

**Definition 3.** Given  $\tau = (t_1, \dots, t_n) \in \mathbb{N}^n$ , define  $A_\tau = R^{[m]}[x^{t_1}z_1, \dots, x^{t_n}z_n]$ . We also set  $A_\tau[\hat{z}_k] = A_\tau \cap R^{[m+n]}[\hat{z}_k] = R^{[m]}[x^{t_1}z_1, \dots, x^{t_k}z_k, \dots, x^{t_n}z_n]$ .

Given  $\tau \in \mathbb{N}^n$  and  $\phi \in \text{GA}_n(R^{[m]})$ , we will consider the natural action

$$\phi^\tau := (x^{-t_1}z_1, \dots, x^{-t_n}z_n) \circ \phi \circ (x^{t_1}z_1, \dots, x^{t_n}z_n)$$

Note that algebraically, the image of this action this gives us the group  $\text{Aut}_{R^{[m]}} A_\tau$ ; we denote the corresponding automorphism group of  $\text{Spec } A_\tau$  by  $\text{GA}_n^\tau(R^{[m]}) \leq \text{GA}_n(S^{[m]})$ . For any subgroup  $H \leq \text{GA}_n(R^{[m]})$ ,

we analogously define  $H^\tau = \{\phi^\tau \mid \phi \in H\} \leq \text{GA}_n^\tau(R^{[m]})$ . We will concern ourselves mostly with  $\text{EA}_n^\tau(R^{[m]})$ ,  $\text{GL}_n^\tau(R^{[m]})$ ,  $\text{GP}_n^\tau(R^{[m]})$ , and  $\text{IA}_n^\tau(R^{[m]})$ .

We also define, choosing variables  $R[y_1, \dots, y_m] = R^{[m]}$ ,

$$\text{IA}_{m+n}^\tau(R) := \text{IA}_{m+n}^{(\mathbf{0}, \tau)}(R) = \{(y_1, \dots, y_m, x^{-t_1}z_1, \dots, x^{-t_n}z_n) \circ \phi \circ (y_1, \dots, y_m, x^{t_1}z_1, \dots, x^{t_n}z_n) \mid \phi \in \text{IA}_{m+n}(R)\}$$

where  $(\mathbf{0}, \tau) = (0, \dots, 0, t_1, \dots, t_n) \in \mathbb{N}^{m+n}$ . Note that  $\text{IA}_{m+n}^\tau(R) \subset \text{IA}_n^\tau(R^{[m]})$ .

Automorphisms in these subgroups can be characterized by the following lemma.

**Lemma 2.** *Let  $\tau = (t_1, \dots, t_n) \in \mathbb{N}^n$ .*

1. *Let  $\alpha \in \text{IA}_{m+n}^\tau(R)$ . Then there exist  $F_1, \dots, F_m, G_1, \dots, G_n \in A_\tau$  such that*

$$\alpha = (y_1 + xF_1, \dots, y_m + xF_m, z_1 + x^{-t_1+1}G_1, \dots, z_n + x^{-t_n+1}G_n)$$

2. *Let  $\Phi \in \text{EA}_n^\tau(R^{[m]})$  be elementary. Then there exists  $P(\hat{z}_k) \in A_\tau[\hat{z}_k]$  such that*

$$\Phi = (z_1, \dots, z_{k-1}, z_k + x^{-t_k}P(\hat{z}_k), z_{k+1}, \dots, z_n)$$

3. *Let  $\gamma \in \text{GL}_n^\tau(R^{[m]})$ . Then there exists  $a_{ij} \in R^{[m]} \setminus xR^{[m]}$  such that*

$$\gamma = (a_{11}z_1 + a_{12}x^{t_2-t_1}z_2 + \dots + a_{1n}x^{t_n-t_1}z_n, \dots, a_{1n}x^{t_1-t_n}z_1 + \dots + a_{n-1,n}x^{t_{n-1}-t_n}z_{n-1} + a_{nn}z_n)$$

The rest of the paper is organized as follows: the most general form of our results is given in Main Theorems 1 and 2 in the next section. Here, we state a couple of less technical versions that are easier to apply. This section concludes with some more concrete applications of these results. The subsequent section consists of a series of increasingly technical lemmas culminating in the two Main Theorems in section 3.2.

**Theorem 3.** *Let  $\phi \in \text{EA}_n(S^{[m]})$ , and write  $\phi = \Phi_0 \circ \dots \circ \Phi_q$  as a product of elementaries. For  $0 \leq i \leq q$  define  $\tau_i \in \mathbb{N}^n$  to be minimal such that  $(\Phi_i \circ \dots \circ \Phi_q)(A_{\tau_i}) \subset R^{[m+n]}$ . Let  $\alpha \in \text{IA}_{n+m}^{\tau_0}(R)$ , and set  $\theta = \alpha \circ \phi$ . Suppose also that either*

1. *A is an integral domain of characteristic zero and  $n = 2$ , or*

2.  *$\Phi_i \in \text{EA}_n^{\tau_i}(R^{[m]})$  for  $0 \leq i \leq q$*

*Then  $(\theta(y_1), \dots, \theta(y_m))$  form a partial system of coordinates over  $R$ . Moreover, if  $A$  is a regular domain and  $\alpha \in \text{TA}_{m+n}(S)$ , then  $(\theta(y_1), \dots, \theta(y_m))$  can be extended to a stably tame automorphism over  $R$ .*

*Proof.* If we assume hypothesis 1, the theorem follows immediately from Main Theorem 2. If we instead assume the second hypothesis, we need only to show that  $\tau_0 \geq \dots \geq \tau_q$ , as then the result follows from Main Theorem 1. Let  $i < q$ . Since  $\Phi_i \in \text{EA}_n^{\tau_i}(R^{[m]})$ , we have  $\Phi_i(A_{\tau_i}) = A_{\tau_i}$ . Then  $(\Phi_i \circ \dots \circ \Phi_q)(A_{\tau_i}) = (\Phi_{i+1} \circ \dots \circ \Phi_q)(A_{\tau_i}) \subset R^{[m+n]}$ . Then the minimality assumption on  $\tau_{i+1}$  immediately implies  $\tau_i \geq \tau_{i+1}$  as required.  $\square$

It is often more practical to rephrase the general ( $n > 2$ ) case in the following way:

**Theorem 4.** *Let  $\phi \in \text{EA}_n(S^{[m]})$ , and write  $\phi = \Phi_0 \circ \dots \circ \Phi_q$  as a product of elementaries. Set  $\sigma_{q+1} = \mathbf{0} \in \mathbb{N}^n$ , and for  $0 \leq i \leq q$  define  $\sigma_i \in \mathbb{N}^n$  to be minimal such that  $\Phi_i(A_{\sigma_i}) \subset A_{\sigma_{i+1}}$ . Let  $\alpha \in \text{IA}_{n+m}^{\sigma_0}(R)$ , and set  $\theta = \alpha \circ \phi$ . Then  $(\theta(y_1), \dots, \theta(y_m))$  is a partial system of coordinates over  $R$ . Moreover, if  $A$  is a regular domain and  $\alpha \in \text{TA}_{m+n}(S)$ , then  $(\theta(y_1), \dots, \theta(y_m))$  can be extended to a stably tame automorphism over  $R$ .*

*Proof.* The following two facts are immediate from the definition of  $\sigma_i$ :

1.  $\Phi_i \in \text{EA}_n^{\sigma_i}(R^{[m]})$

2.  $\sigma_0 \geq \dots \geq \sigma_q$

Once these are shown, we can apply Main Theorem 1 to achieve the result. To see these two facts, write  $\sigma_i = (s_{i,1}, \dots, s_{i,n})$ . Without loss of generality, suppose  $\Phi_i$  is elementary in  $z_1$ , and write

$$\Phi_i = (z_1 + x^{-s}P(x^{s_{i+1,2}}z_2, \dots, x^{s_{i+1,n}}z_n), z_2, \dots, z_n)$$

for some  $P(\hat{z}_1) \in A_{\sigma_{i+1}}[\hat{z}_1] \setminus xA_{\sigma_{i+1}}[\hat{z}_1]$ . Clearly, the minimality condition on  $\sigma_i$  guarantees  $s_{i,k} = s_{i+1,k}$  for  $k = 2, \dots, n$ . Since  $\Phi_i(x^{s_{i,1}}z_1) = x^{s_{i,1}}z_1 + x^{s_{i,1}-s}P(\hat{z}_1) \in A_{\sigma_{i+1}} \setminus xA_{\sigma_{i+1}}$ , we see  $s_{i,1} \geq s_{i+1,1}$  (giving  $\sigma_i \geq \sigma_{i+1}$ ) and  $s \leq s_{i,1}$ . From the latter, one easily sees that  $\Phi_i = (x^{-s_{i,1}}z_1, \dots, x^{-s_{i,n}}z_n) \circ (z_1 + x^{s_{i,1}-s}P(z_2, \dots, z_n), z_2, \dots, z_n) \circ (x^{s_{i,1}}z_1, \dots, x^{s_{i,n}}z_n) \in \text{EA}_n^{\sigma_i}(R^{[m]})$ .  $\square$

The remainder of this section is devoted to consequences of these three theorems in more concrete settings.

*Example 1.* Let  $m = 1$  and  $n = 1$ . Set

$$\alpha = (y + x^2z, z) \quad \Phi_0 = (y, z - \frac{y^2}{x})$$

Theorem 4 implies  $(\alpha \circ \Phi_0)(y) = y + x(xz - y^2)$  is a coordinate. The construction produces the Nagata map

$$\sigma = (y + x(xz - y^2), z + 2y(xz - y^2) + x(xz - y^2)^2)$$

*Example 2.* Let  $m = 1$ ,  $n = 1$ , and  $R = k[x, t]$ . Set

$$\alpha = (y + x^2z, z) \quad \Phi_0 = (y, z + \frac{yt}{x})$$

Theorem 4 implies  $y + x(xz + yt)$  is a coordinate. The construction produces Anick's example

$$\beta = (y + x(xz + yt), z - t(xz + yt))$$

In [11], a generalization of the Vénéreau polynomial called Vénéreau-type polynomials were studied by the author. They are polynomials of the form  $y + xQ(xz + y(yu + z^2), x^2u - 2xz(yu + z^2) - y(yu + z^2)^2) \in \mathbb{C}[x, y, z, u]$  where  $Q \in \mathbb{C}[x]^{[2]}$ . Many Vénéreau-type polynomials remain as strongly residual coordinates that have not been resolved as coordinates. However, we are able to show them all to be 1-stable coordinates, generalizing Freudenburg's result [6] that the Vénéreau polynomial is a 1-stable coordinate<sup>1</sup>.

**Corollary 5.** *Every Vénéreau-type polynomial is a 1-stable coordinate.*

*Proof.* Let  $Q \in \mathbb{C}[x][xz, x^2u]$ , and set

$$\begin{aligned} \alpha &= (y + xQ, z, u, t) \\ \Phi_0 &= (y, z + yt, u, t) & \Phi_3 &= (y, z - yt, u, t) \\ \Phi_1 &= (y, z, u - 2zt - yt^2, t) & \Phi_4 &= (y, z, u - 2zt + yt^2) \\ \Phi_2 &= (y, z, u, t + \frac{yu + z^2}{x}) \end{aligned}$$

A direct computation shows that  $(\alpha \circ \Phi_0 \circ \dots \circ \Phi_4)(y) = y + xQ(xz + y(yu + z^2), x^2u - 2xz(yu + z^2) - y(yu + z^2)^2)$  is an arbitrary Vénéreau-type polynomial. We compute the induced  $\sigma$ -sequence  $(1, 2, 1) \geq (0, 2, 1) \geq (0, 0, 1) \geq (0, 0, 0) \geq (0, 0, 0)$ , and note that since  $Q \in A_{\sigma_0}$ , then  $\alpha_0 \in \text{IA}_4^{\sigma_0}(\mathbb{C}[x])$ . It then follows immediately from Theorem 4 that any Vénéreau-type polynomial is a  $\mathbb{C}[x]$  coordinate in  $\mathbb{C}[x][y, z, u, t]$ .  $\square$

The following result is first due to Russell [12], and later appeared also in [4].

**Corollary 6.** *Let  $k$  be a field, and let  $P \in k[x, y, z]$  be of the form  $P = y + xf(x, y) + \lambda x^s z$  for some  $s \in \mathbb{N}$ ,  $\lambda \in k^*$  and  $f \in k[x, y]$ . Then  $P$  is a  $k[x]$ -coordinate.*

*Proof.* Here  $R = k[x]$  and  $S = k[x, x^{-1}]$ . Let  $\theta = (y + \lambda x^s z, z) \circ (y, z + \lambda^{-1}x^{1-s}f(x, y)) \in \text{EA}_3(S)$ . Then Theorem 3 yields  $\theta(y)$  is a  $k[x]$ -coordinate, and one easily checks that  $\theta(y) = P$ .  $\square$

<sup>1</sup>Our construction provides a different coordinate system than Freudenburg's.

### 3 Main results

Due to the tedious nature of some of these calculations, the reader is advised to first simply read the statements of the results in section 3.1, and return for the details after reading the proofs of the main theorems (section 3.2).

#### 3.1 Calculations

We proceed by detailing a series of (increasingly technical) lemmas that will aid in the proofs of the main theorems. First, a straightforward application of Taylor's formula yields the following.

**Lemma 7.** *Let  $\tau \in \mathbb{N}^n$ ,  $P \in A_\tau$ .*

1. *If  $\phi \in \text{GA}_n^\tau(R^{[m]})$ , then  $\phi(A_\tau) = A_\tau$ .*
2. *If  $\alpha \in \text{IA}_{m+n}^\tau(R)$ , then  $\alpha(P) - P \in xA_\tau$ .*

Next, we note that  $\text{GA}_n^\tau(R^{[m]})$  is contained in the normalizer of  $\text{IA}_{m+n}^\tau(R)$  in  $\text{GA}_{m+n}(S)$ . This is slightly more general than the fact that  $\text{IA}_n^\tau(R^{[m]}) \triangleleft \text{GA}_n^\tau(R^{[m]})$ .

**Lemma 8.** *Let  $\tau \in \mathbb{N}^n$ . Then  $\text{IA}_{m+n}^{(0,\tau)}(R) \triangleleft \text{GA}_{m+n}^{(0,\tau)}(R)$ . In particular, for any  $\alpha \in \text{IA}_{m+n}^\tau(R)$  and  $\phi \in \text{GA}_n^\tau(R^{[m]})$ , we have  $\phi^{-1} \circ \alpha \circ \phi \in \text{IA}_{m+n}^\tau(R)$ .*

*Proof.* Simply note that the surjection  $R = A[x] \rightarrow A$  induces a short exact sequence

$$0 \rightarrow \text{IA}_{m+n}^{(0,\tau)}(R) \rightarrow \text{GA}_{m+n}^{(0,\tau)}(R) \rightarrow \text{GA}_{m+n}^{(0,\tau)}(A) \rightarrow 0 \quad (2)$$

Here, we are viewing  $\text{GA}_{m+n}(A) \leq \text{GA}_{m+n}(R)$  by extension of scalars, and thus obtaining  $\text{GA}_{m+n}^{(0,\tau)}A \leq \text{GA}_{m+n}^{(0,\tau)}A$ .  $\square$

**Corollary 9.** *Let  $\tau \in \mathbb{N}^n$ ,  $\alpha \in \text{IA}_{m+n}^\tau(R)$ , and  $\phi \in \text{GA}_n^\tau(R^{[m]})$ . Then there exists  $\alpha' \in \text{IA}_{m+n}^\tau(R)$  such that  $\alpha \circ \phi = \phi \circ \alpha'$ .*

**Lemma 10.** *Let  $\tau \in \mathbb{N}^n$  and  $\alpha \in \text{IA}_{m+n}^\tau(R)$ . Then there exists  $\phi \in \text{EA}_n^\tau(R^{[m]}) \cap \text{IA}_n^\tau(R^{[m]})$  such that*

$$\phi \circ \alpha \in \bigcap_{0 \leq \sigma \leq \tau} \text{IA}_{m+n}^\sigma(R)$$

*Proof.* We begin by writing  $\tau = (t_1, \dots, t_n) \in \mathbb{N}^n$  and

$$\alpha = (y_1 + xF_1, \dots, y_m + xF_m, z_1 + x^{-t_1+1}Q_1, \dots, z_n + x^{-t_n+1}Q_n) \quad (3)$$

for some  $F_1, \dots, F_m, Q_1, \dots, Q_n \in A_\tau$ . We prove the following by induction.

**Claim 11.** *For any  $\sigma' = (s_1, \dots, s_n) \in \mathbb{N}^n$ , there exists  $\phi \in \text{EA}_n^\tau(R^{[m]}) \cap \text{IA}_n^\tau(R^{[m]})$  such that  $\phi \circ \alpha \in \text{IA}_{m+n}^{\sigma'}(R)$  is of the form*

$$\phi \circ \alpha = (y_1 + xF_1, \dots, y_m + xF_m, z_1 + xz_1G_1 + x^{-t_1+s_1+1}H_1, \dots, z_n + xz_nG_n + x^{-t_n+s_n+1}H_n)$$

*for some  $G_1, H_1, \dots, G_n, H_n \in A_\tau$ .*

Clearly the case  $\sigma' = \tau$  proves the lemma, since  $\sigma \leq \tau$  implies  $A_\tau \subset A_\sigma$ . We induct on  $\sigma'$  in the partial ordering of  $\mathbb{N}^n$ . Our base case of  $\sigma' = (0, \dots, 0)$  is provided by  $\phi = \text{id}$  (from (3)). So we assume  $\sigma' > 0$ .

Suppose the claim holds for  $\sigma'$ . We will show that this implies the claim for  $\sigma' + e_k$ , where  $e_k$  is the  $k$ -th standard basis vector of  $\mathbb{N}^n$ . Without loss of generality, we take  $k = 1$ , so  $e_1 = (1, 0, \dots, 0)$ . By the inductive hypothesis, we may write

$$\alpha' := \phi \circ \alpha = (y_1 + xF_1, \dots, y_m + xF_m, z_1 + xz_1G_1 + x^{-t_1+s_1+1}H_1, \dots, z_n + xz_nG_n + x^{-t_n+s_n+1}H_n)$$

for some  $G_i, H_i \in A_\tau$  and  $\phi \in \text{EA}_n^\tau(R^{[m]}) \cap \text{IA}_n^\tau(R^{[m]})$ . Write  $H_1 = P(\hat{z}_1) + x^{t_1} z_1 Q$  for some  $Q \in A_\tau$  and  $P(\hat{z}_1) \in A_\tau[\hat{z}_1]$ . Then we may set  $\phi' = (z_1 - x^{-t_1+s_1+1}P(\hat{z}_1), z_2, \dots, z_n) \in \text{EA}_n^\tau(R^{[m]}) \cap \text{IA}_n^\tau(R^{[m]})$  and compute

$$\begin{aligned} (\phi' \circ \alpha')(z_1) &= z_1 + xz_1G_1 + x^{-t_1+s_1+1}(H_1 - \alpha'(P(\hat{z}_1))) \\ &= z_1 + xz_1G_1 + x^{-t_1+s_1+1}(P(\hat{z}_1) + x^{t_1}z_1Q - \alpha'(P(\hat{z}_1))) \\ &= z_1 + xz_1(G_1 + x^{s_1}Q) + x^{-t_1+s_1+1}(P(\hat{z}_1) - \alpha'(P(\hat{z}_1))) \end{aligned} \quad (4)$$

Since  $\alpha' \in \text{IA}_{m+n}^\tau(R)$ , we can write (by Lemma 7)  $\alpha'(P(\hat{z}_1)) = P(\hat{z}_1) - xH'_1$  for some  $H'_1 \in A_\tau$ . We also set  $G'_1 = G_1 + x^{s_1}Q \in A_\tau$ , and thus clearly see from (4) that  $\phi' \circ \alpha'$  is of the required form:

$$\begin{aligned} \phi' \circ \alpha' &= (y_1 + xF_1, \dots, y_m + xF_m, z_1 + xz_1G'_1 + x^{-t_1+(s_1+1)+1}H'_1, \\ &\quad z_2 + xz_2G_2 + x^{-t_2+s_2+1}H_2, \dots, z_n + xz_nG_n + x^{-t_n+s_n+1}H_n) \end{aligned}$$

□

**Corollary 12.** *Let  $\sigma \leq \tau \in \mathbb{N}^n$ , and let  $\alpha \in \text{IA}_{m+n}^\tau(R)$ . Then there exists  $\beta \in \text{IA}_{m+n}^\sigma(R^{[m]})$  and  $\phi \in \text{EA}_n^\tau(R^{[m]})$  such that  $\alpha = \beta \circ \phi$ . Moreover, if  $\tau - \sigma = (0, \dots, 0, \delta, 0, \dots, 0)$  then  $\phi$  can be taken to be elementary.*

**Theorem 13.** *Let  $\tau \in \mathbb{N}^n$ ,  $\alpha \in \text{IA}_{m+n}^\tau(R)$ , and let  $\phi \in \text{GA}_n^\tau(R^{[m]})$ . Then there exists  $\tilde{\phi} \in \langle \phi, \text{EA}_n^\tau(R^{[m]}) \rangle$  such that*

$$\tilde{\phi} \circ \alpha \circ \phi \in \bigcap_{\mathbf{0} \leq \sigma \leq \tau} \text{IA}_{m+n}^\sigma(R)$$

*In particular,  $\tilde{\phi} \circ \alpha \circ \phi \in \text{IA}_{m+n}(R)$ ; and if  $\phi, \alpha \in \text{TA}_{m+n}(S)$ , then  $\tilde{\phi} \circ \alpha \circ \phi \in \text{TA}_{m+n}(S)$  as well.*

*Proof.* By Lemma 8,  $\phi^{-1} \circ \alpha \circ \phi \in \text{IA}_{m+n}^\tau(R)$ . But then by Lemma 10, there exists  $\psi \in \text{EA}_n^\tau(R^{[m]})$  such that  $\psi \circ (\phi^{-1} \circ \alpha \circ \phi) \in \bigcap_{\mathbf{0} \leq \sigma \leq \tau} \text{IA}_{m+n}^\sigma(R)$ . So we simply set  $\tilde{\phi} = \psi \circ \phi^{-1} \in \langle \phi, \text{EA}_n^\tau(R^{[m]}) \rangle$  to obtain the desired result. □

At this point, one could go ahead and directly prove Main Theorem 1. However, it will be useful in proving Main Theorem 2 to have the stronger result of Theorem 27 (which immediately implies Main Theorem 1). To prove Theorem 27, we need to study  $\text{GP}_n(S^{[m]})$  and its relation with  $\text{IA}_{m+n}^\tau(R)$ .

**Definition 4.** Let  $A$  be a connected, reduced ring. Given  $\rho \in \text{GP}_n(S^{[m]})$ , we can then write  $\rho = (\lambda_{\sigma(1)}x^{r_{\sigma(1)}}z_{\sigma(1)}, \dots, \lambda_{\sigma(n)}x^{r_{\sigma(n)}}z_{\sigma(n)})$  for some permutation  $\sigma \in \mathfrak{S}_n$ ,  $\lambda_i \in A^*$ , and  $r_i \in \mathbb{Z}$ . If we are also given  $\tau = (t_1, \dots, t_n) \in \mathbb{N}^n$ , we can define  $\rho(\tau) = (t_{\sigma^{-1}(1)} + r_1, \dots, t_{\sigma^{-1}(n)} + r_n) \in \mathbb{Z}^n$ .

*Remark 4.* The condition that  $A$  is connected and reduced is essential to obtain  $(A^{[m]}[x, x^{-1}])^* = \{\lambda x^r \mid \lambda \in A^*, r \in \mathbb{Z}\}$ , which is what allows us to write  $\rho$  in the given form.

The definition of  $\rho(\tau)$  is chosen precisely so that the following lemma holds.

**Lemma 14.** *Let  $A$  be a connected, reduced ring, let  $\rho \in \text{GP}_n(S^{[m]})$  and let  $\tau \in \mathbb{N}^n$ . Then  $\rho(A_\tau) = A_{\rho(\tau)}$ .*

Recall that for  $\tau = (t_1, \dots, t_n) \in \mathbb{N}^n$ , one obtains  $\phi^\tau$  from  $\phi$  via conjugation by  $(x^{t_1}z_1, \dots, x^{t_n}z_n) \in \text{GP}_n(S^{[m]})$ . Then, recalling that for any subgroup  $H \leq \text{GA}_n(R^{[m]})$ ,  $H^\tau = \{\phi^\tau \mid \phi \in H\}$ , we immediately see the following.

**Lemma 15.** *Let  $H \leq \text{GA}_n(R^{[m]})$ ,  $\tau \in \mathbb{N}^n$ , and  $\rho \in \text{GP}_n(S^{[m]})$ . Let  $\phi \in \text{GA}_n(S^{[m]})$ . Then  $\phi \in H^\tau$  if and only if  $\rho^{-1} \circ \Phi \circ \rho \in H^{\rho(\tau)}$ .*

**Corollary 16.** *Let  $A$  be a connected, reduced ring, let  $\tau \in \mathbb{N}^n$ , and let  $\alpha \in \text{IA}_{m+n}^\tau(R)$  and  $\rho \in \text{GP}_n(S^{[m]})$ . Then  $\rho^{-1} \circ \alpha \circ \rho \in \text{IA}_{m+n}^{\rho(\tau)}(R)$ .*

**Corollary 17.** *Let  $A$  be a connected, reduced ring, let  $\tau \in \mathbb{N}^n$ , let  $\Phi \in \text{EA}_n^\tau(R^{[m]})$  be elementary, and let  $\rho \in \text{GP}_n(S^{[m]})$ . Then  $\Phi \in \text{EA}_n^{\rho(\tau)}(R^{[m]})$  if and only if  $\rho \circ \Phi \circ \rho^{-1} \in \text{EA}_n^\tau(R^{[m]})$ .*

**Corollary 18.** *Let  $A$  be a connected, reduced ring, let  $\phi \in \text{EA}_n^\tau(R^{[m]})$  and  $\rho \in \text{GP}_n(S^{[m]})$ . Then there exists  $\phi' \in \text{EA}_n^{\rho(\tau)}(R^{[m]})$  such that  $\phi \circ \rho = \rho \circ \phi'$ . Moreover, if  $\phi$  is elementary, then so is  $\phi'$ .*

We now have the necessary tools to prove Theorem 27, and the interested reader may skip ahead. The rest of this section develops some more tools for use in the proof of Main Theorem 2.

**Lemma 19.** *Let  $A$  be a connected, reduced ring. Let  $\tau_0, \dots, \tau_{q+1} \in \mathbb{N}^n$ . Let  $\rho_i \in \text{GP}_n(S^{[m]})$ ,  $\alpha_i \in \text{IA}_{m+n}^{\tau_i}(R)$ , and  $\phi_i \in \text{EA}_n^{\tau_{i+1}}(R^{[m]})$  for  $0 \leq i \leq q$ . Suppose also that  $\rho_i(\tau_i) \leq \tau_{i+1}$  for each  $0 \leq i \leq q$ . Then there exist  $\alpha' \in \text{IA}_{m+n}^{\tau_0}(R)$  and, for each  $0 \leq i < q$ ,  $\phi'_i \in \text{EA}_n^{(\rho_{i+1} \circ \dots \circ \rho_q)(\tau_{i+1})}(R^{[m]})$  such that*

$$\alpha_0 \circ \rho_0 \circ \phi_0 \circ \dots \circ \alpha_q \circ \rho_q \circ \phi_q = \alpha' \circ (\rho_0 \circ \dots \circ \rho_q) \circ \phi'_0 \circ \phi'_1 \circ \dots \circ \phi'_{q-1} \circ \phi_q$$

*Proof.* We induct on  $q$ . If  $q = 0$ , the claim is trivial, so assume  $q > 0$ . So by the inductive hypothesis, we may assume

$$\alpha_1 \circ \rho_1 \circ \phi_1 \circ \dots \circ \alpha_q \circ \rho_q \circ \phi_q = \alpha'_1 \circ \rho'_1 \circ \phi'_1 \circ \dots \circ \phi'_{q-1} \circ \phi_q$$

for some  $\alpha'_1 \in \text{IA}_{m+n}^{\tau_1}(R)$ ,  $\phi'_i \in \text{EA}_n^{(\rho_{i+1} \circ \dots \circ \rho_q)(\tau_{i+1})}(R^{[m]})$ , and  $\rho'_1 = \rho_1 \circ \dots \circ \rho_q$ . Note that it now suffices to find  $\alpha' \in \text{IA}_n^{\tau_0}(R^{[m]})$  and  $\phi'_0 \in \text{EA}_2^{\rho'_1(\tau_1)}(R^{[m]})$  such that

$$\alpha_0 \circ \rho_0 \circ \phi_0 \circ \alpha'_1 \circ \rho'_1 = \alpha' \circ (\rho_0 \circ \rho'_1) \circ \phi'_0$$

From Corollary 9, there exists  $\tilde{\alpha} \in \text{IA}_{m+n}^{\tau_1}(R)$  such that  $\phi_0 \circ \alpha'_1 = \tilde{\alpha} \circ \phi_0$ . By Corollary 16, there exists  $\alpha'' \in \text{IA}_{m+n}^{\rho_0^{-1}(\tau_1)}(R)$  such that  $\rho_0 \circ \tilde{\alpha} = \alpha'' \circ \rho_0$ . In addition, by Corollary 12, since  $\tau_0 \leq \rho_0^{-1}(\tau_1)$ , there exist  $\beta \in \text{IA}_{m+n}^{\tau_0}(R)$  and  $\tilde{\phi} \in \text{EA}_n^{\rho_0^{-1}(\tau_1)}(R^{[m]})$  such that  $\alpha'' = \beta \circ \tilde{\phi}$ . Then we have

$$\alpha_0 \circ \rho_0 \circ \phi_0 \circ \alpha'_1 \circ \rho'_1 = \alpha_0 \circ \rho_0 \circ \tilde{\alpha} \circ \phi_0 \circ \rho'_1 = \alpha_0 \circ \alpha'' \circ \rho_0 \circ \phi_0 \circ \rho'_1 = \alpha_0 \circ \beta \circ \tilde{\phi} \circ \rho_0 \circ \phi_0 \circ \rho'_1 \quad (5)$$

Now by Corollary 18, there exists  $\phi' \in \text{EA}_2^{\tau_1}(R^{[m]})$  such that  $\tilde{\phi} \circ \rho_0 = \rho_0 \circ \phi'$ . Also, there exist  $\phi'_0 \in \text{EA}_2^{\rho'_1(\tau_1)}(R)$  such that  $(\phi' \circ \phi_0) \circ \rho'_1 = \rho'_1 \circ \phi'_0$ . Then from (5), we obtain

$$\alpha_0 \circ \beta \circ \tilde{\phi} \circ \rho_0 \circ \phi_0 \circ \alpha'_1 \circ \rho'_1 = \alpha_0 \circ \beta \circ \rho_0 \circ (\phi' \circ \phi_0) \circ \rho'_1 = \alpha_0 \circ \beta \circ \rho_0 \circ \rho'_1 \circ \phi'_0$$

Now we simply set  $\alpha' = \alpha_0 \circ \beta \in \text{IA}_{m+n}^{\tau_0}(R)$  to achieve the desired result.  $\square$

In fact, this same proof gives the following, noting that the hypothesis  $\tau_2 - \rho_1(\tau_1) = \delta e_k$  is what implies that the resulting  $\Phi$  is elementary:

**Corollary 20.** *Let  $A$  be a connected, reduced ring. Suppose  $\tau_1, \tau_2 \in \mathbb{N}^n$ ,  $\alpha_1 \in \text{IA}_{m+n}^{\tau_1}(R)$ ,  $\alpha_2 \in \text{IA}_{m+n}^{\tau_2}(R)$ , and  $\rho_1, \rho_2 \in \text{GP}_2(S^{[m]})$ . If  $\tau_2 - \rho_1(\tau_1) = \delta e_k$  for some  $1 \leq k \leq n$  and  $\delta \in \mathbb{N}$ , then there exist  $\alpha' \in \text{IA}_{m+n}^{\tau_1}(R)$ ,  $\rho' \in \text{GP}_2(S^{[m]})$ , and elementary  $\Phi \in \text{EA}_n^{\rho_2(\tau_2)}(R^{[m]})$  such that*

$$\alpha_1 \circ \rho_1 \circ \alpha_2 \circ \rho_2 = \alpha' \circ \rho' \circ \Phi$$

We alert the reader to the fact that the next lemma is true only for  $n = 2$ .

**Lemma 21.** *Assume  $A$  is an integral domain of characteristic zero. Let  $\sigma \leq \tau = (t_1, t_2) \in \mathbb{N}^2$ , let  $\Phi \in \text{EA}_2(S^{[m]})$  be elementary, and let  $\omega \in \text{GA}_{m+2}(S)$  such that  $\omega(x^{t_1} z_1), \omega(x^{t_2} z_2) \in R^{[m+2]} \setminus xR^{[m+2]}$ . If  $(\Phi \circ \omega)(A_\sigma) \subset R^{[m+2]}$ , then  $\Phi \in \text{EA}_2^\tau(R^{[m]})$ .*

*Proof.* Without loss of generality, assume that  $\Phi$  is elementary in  $z_1$ . Then setting  $\sigma = (s_1, s_2)$ , it is clear that  $s_2 = t_2$ . Write  $\Phi = (z_1 + x^{-r} P(x^{t_2} z_2), z_2)$  for some  $P \in A_\tau[\hat{z}_1] \setminus xA_\tau[\hat{z}_1]$ . It suffices to show that  $r \leq t_1$ . We compute

$$(\Phi \circ \omega)(x^{t_1} z_1) = \omega(x^{t_1} z_1) + x^{t_1-r} P(\omega(x^{t_2} z_2))$$

Since  $\sigma \leq \tau$  and  $(\Phi \circ \omega)(A_\sigma) \in R^{[m+2]}$ , we must have  $(\Phi \circ \omega)(x^{t_1} z_1) \in R^{[m+2]}$ . But  $\omega(x^{t_1} z_1) \in R^{[m+2]}$  by assumption, so we then have  $x^{t_1-r} P(\omega(x^{t_2} z_2)) \in R^{[m+2]}$ . Since  $P \notin (x)$  and  $\omega(x^{t_2} z_2) \in R^{[m+2]} \setminus xR^{[m+2]}$ , we thus must have  $r \leq t_1$  as required.  $\square$



We remark that if  $n = 3$ , one may find  $P \notin (x)$  and  $\omega$  with  $\omega(x^{t_2}z_2), \omega(x^{t_3}z_3) \in R^{[m+2]} \setminus xR^{[m+2]}$ , but  $P(\omega(x^{t_2}z_2), \omega(x^{t_3}z_3)) \in xR^{[m+2]}$ . For example, set  $\tau = (0, 1, 2)$ ,  $\omega = (z_1, z_2 - \frac{yz_1}{x}, z_3 - \frac{2z_2z_1}{x} - \frac{yz_1^2}{x^2}) \in \text{GA}_3(\mathbb{C}[x, x^{-1}][y])$ , and let  $P = y(x^2z_3) + (xz_2)^2 \in A_\tau[\hat{z}_1] \setminus xA_\tau[\hat{z}_1]$ . Then one easily checks that  $P(\omega(xz_2), \omega(x^2z_3)) = x^2(yz_3 + z_2^2) \in (x^2)\mathbb{C}[x][y]^{[3]}$ . This type of behavior is a crucial difficulty in extending our result from the  $n = 2$  case to  $n \geq 3$ . Additionally, if we relax our hypotheses on the ring  $A$ , we create similar difficulties.

**Lemma 22.** *Let  $\tau = (t_1, t_2) \in \mathbb{N}^2$ . Let  $\Phi \in \text{EA}_2^\tau(R^{[m]})$  be elementary, and let  $\beta = (az_1 + bx^{t_2-t_1}z_2, dz_2 + cx^{t_1-t_2}z_1) \in \text{GL}_2^\tau(R^{[m]})$ .*

1. *If  $\Phi$  is elementary in  $z_1$  and either  $c = 0$  or  $d = 0$ , then there exists  $\rho \in \text{GP}_2^\tau(R^{[m]})$  and elementary  $\Phi' \in \text{EA}_2^\tau(R^{[m]})$  such that  $\Phi \circ \beta = \rho \circ \Phi'$ .*
2. *If  $\Phi$  is elementary in  $z_2$  and either  $a = 0$  or  $b = 0$ , then there exists  $\rho \in \text{GP}_2^\tau(R^{[m]})$  and elementary  $\Phi' \in \text{EA}_2^\tau(R^{[m]})$  such that  $\Phi \circ \beta = \rho \circ \Phi'$ .*

*Proof.* Suppose  $\Phi$  is elementary in  $z_1$  and write  $\Phi = (z_1 + x^{-t_1}P(x^{t_2}z_2), z_2)$ . First, suppose  $c = 0$ , so

$$\Phi \circ \beta = (az_1 + bx^{t_2-t_1}z_2 + x^{-t_1}P(dx^{t_2}z_2), dz_2) = (az_1, dz_2) \circ (z_1 + \frac{1}{a}x^{-t_1}(bx^{t_2}z_2 + P(dx^{t_2}z_2)), z_2)$$

If instead  $d = 0$ , then

$$\Phi \circ \beta = (az_1 + bx^{t_2-t_1}z_2 + x^{-t_1}P(cx^{t_1}z_1), cx^{t_1-t_2}z_1) = (bx^{t_2-t_1}z_2, cx^{t_1-t_2}z_1) \circ (z_1, z_2 + \frac{1}{b}x^{-t_2}(ax^{t_1}z_1 + P(cx^{t_1}z_1)), z_2)$$

These are both precisely in the desired form. The case where  $\Phi$  is elementary in  $z_2$  follows similarly.  $\square$

We conclude with two technical lemmas.

**Lemma 23.** *Suppose  $A$  is a connected, reduced ring. Let  $\tau_0, \dots, \tau_q \in \mathbb{N}^n$ ,  $\Phi_0, \dots, \Phi_q \in \text{EA}_n(S^{[m]})$  be elementaries,  $\alpha_i \in \text{IA}_{m+n}^{\tau_i}(R)$ , and  $\rho_i \in \text{GP}_n(S^{[m]})$ . Set*

$$\omega_i = \alpha_i \circ \rho_i \circ \Phi_i \circ \dots \circ \alpha_q \circ \rho_q \circ \Phi_q$$

*Also set  $\Phi'_i = \rho_i \circ \Phi_i \circ \rho_i^{-1}$ . Then the following conditions are equivalent:*

1. *Each  $\tau_i \in \mathbb{N}^n$  is minimal such that  $\omega_i(A_{\tau_i}) \subset R^{[m+n]}$*
2. *Each  $\tau_i \in \mathbb{N}^n$  is minimal such that  $(\Phi_i \circ \omega_{i+1})(A_{\rho_i(\tau_i)}) \subset R^{[m+n]}$ .*
3. *Each  $\tau_i \in \mathbb{N}^n$  is minimal such that  $(\rho_i \circ \Phi_i \circ \omega_{i+1})(A_{\tau_i}) \subset R^{[m+n]}$ .*
4. *Each  $\tau_i \in \mathbb{N}^n$  is minimal such that  $(\Phi'_i \circ \rho_i \circ \omega_{i+1})(A_{\tau_i}) \subset R^{[m+n]}$ .*

*Moreover, if the above are satisfied, then, writing  $\tau_i = (t_{i,1}, \dots, t_{i,n})$ ,*

1. *If  $\Phi'_i$  is elementary in  $z_j$ , then  $(\rho_i \circ \omega_{i+1})(x^{t_{i,k}}z_k) \in R^{[m+n]} \setminus xR^{[m+n]}$  for all  $k \neq j$ .*
2. *If  $\Phi_i$  is elementary in  $z_j$ , then  $\rho_i(\tau_i) - \tau_{i+1} = \delta_i e_j$  for some  $\delta_i \in \mathbb{Z}$  (recall  $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ ).*

*Proof.* The equivalence of (2) and (3) is immediate from the fact that  $\rho_i(A_{\tau_i}) = A_{\rho_i(\tau_i)}$ . Since  $\alpha_i \in \text{IA}_{m+n}^{\tau_i}(R) \subset \text{GA}_{m+n}^{\tau_i}(R)$ , we have  $\alpha_i(A_{\tau_i}) = A_{\tau_i}$  and thus  $\omega_i(A_{\tau_i}) = (\rho_i \circ \Phi_i \circ \omega_{i+1})(A_{\tau_i})$ , giving the equivalence of (1) and (3). The equivalence of (3) and (4) follows immediately from the definition of  $\Phi'_i$ .

Suppose now that the four conditions are satisfied. Suppose also that  $\Phi'_i$  is elementary in  $z_j$ , so that  $\Phi'_i(z_k) = z_k$  for  $k \neq j$ . Then (4) immediately implies  $(\rho_i \circ \omega_{i+1})(x^{t_{i,k}}z_k) = (\Phi'_i \circ \rho_i \circ \omega_{i+1})(x^{t_{i,k}}z_k) \in R^{[m+n]} \setminus xR^{[m+n]}$ . Now suppose (perhaps instead) that  $\Phi_i$  is elementary in  $z_j$ . Then  $(\Phi_i \circ \omega_{i+1})(x^s z_k) = \omega_{i+1}(x^s z_k)$  for  $k \neq j$ . The minimal  $s$  such that this lies in  $R^{[m+n]}$  is precisely  $t_{i+1,k}$ , so we see from (2) that  $\rho_i(\tau_i) = \tau_{i+1} + \delta_i e_j$  for some  $\delta_i \in \mathbb{Z}$ .  $\square$

We make the following definition to aid in the proof of the Lemma 25.

**Definition 5.** Let  $\tau = (t_1, \dots, t_n) \in \mathbb{N}$ . Note that as in (2), we have  $\text{GA}_n(A^{[m]}) \leq \text{GA}_n(R^{[m]})$ . Then, we can consider  $\text{EA}_n(A^{[m]}) \leq \text{GA}_n(R^{[m]})$ , and define

$$\text{EA}_n^\tau(A^{[m]}) := \{(x^{-t_1}z_1, \dots, x^{-t_n}z_n) \circ \phi \circ (x^{t_1}z_1, \dots, x^{t_n}z_n) \mid \phi \in \text{EA}_n^\tau(A^{[m]})\} \leq \text{GA}_n^\tau(R^{[m]})$$

Given  $\Phi \in \text{EA}_n^\tau(R^{[m]})$ , we will denote its image under the natural quotient as  $\bar{\Phi} \in \text{EA}_n^\tau(A^{[m]})$ .

We can define other subgroups such as  $\text{GL}_2^\tau(A^{[m]})$  in a similar way.

**Lemma 24.** Let  $\tau \in \mathbb{N}$ , and let  $\Phi_1, \dots, \Phi_q \in \text{EA}_n^\tau(R^{[m]})$ . Then there exist  $\alpha \in \text{IA}_n^\tau(R^{[m]})$  and  $\tilde{\Phi}_1, \dots, \tilde{\Phi}_q \in \text{EA}_n^\tau(A^{[m]})$  such that  $\Phi_1 \circ \dots \circ \Phi_q = \alpha \circ \tilde{\Phi}_1 \circ \dots \circ \tilde{\Phi}_q$ .

*Proof.* The key observation is that if  $\Phi \in \text{EA}_n^\tau(R^{[m]})$ , then  $\Phi \circ \bar{\Phi}^{-1} \in \text{IA}_n^\tau(R^{[m]})$ . Then Corollary 9 and a quick induction suffice to prove the lemma.  $\square$

**Lemma 25.** Suppose  $A$  is an integral domain. Let  $\tau = (t_1, t_2) \in \mathbb{N}^2$ , let  $\Phi_1, \dots, \Phi_q \in \text{EA}_2^\tau(R^{[m]})$  be elementaries, and let  $\omega \in \text{GA}_2(S^{[m]})$ . Assume that

1. Either  $\omega(x^{t_1}z_1) \in xR^{[m+2]}$  and  $\omega(x^{t_2}z_2) \in R^{[m+2]} \setminus xR^{[m+2]}$ , or  $\omega(x^{t_2}z_2) \in xR^{[m+2]}$  and  $\omega(x^{t_1}z_1) \in R^{[m+2]} \setminus xR^{[m+2]}$ .
2. Setting  $\omega_i = \Phi_i \circ \dots \circ \Phi_q \circ \omega$ ,  $\omega_i(x^{t_1}z_1), \omega_i(x^{t_2}z_2) \in R^{[m+2]} \setminus xR^{[m+2]}$  for  $1 < i \leq q$
3.  $\omega_1(x^{t_1}z_1) \in xR^{[m+2]}$

Then either all  $\Phi_i$  are elementary in the same variable, or there exists  $\alpha \in \text{IA}_2^\tau(R^{[m]})$ ,  $\rho \in \text{GP}_2^\tau(R^{[m]})$  and elementary  $\Phi \in \text{EA}_2^\tau(R^{[m]}) \cap \text{GL}_2^\tau(R^{[m]})$  such that  $\Phi_1 \circ \dots \circ \Phi_q = \alpha \circ \rho \circ \Phi$ .

*Proof.* Note that it suffices to assume that  $\Phi_i$  and  $\Phi_{i+1}$  are not elementary in the same variable for each  $i$ , and we may assume  $q \geq 2$ . Moreover, by Lemma 24, we may write  $\Phi_1 \circ \dots \circ \Phi_q = \alpha \circ \tilde{\Phi}_1 \circ \dots \circ \tilde{\Phi}_q$ , for some  $\tilde{\Phi}_i \in \text{EA}_2^\tau(A^{[m]})$  and  $\alpha \in \text{IA}_2^\tau(R^{[m]})$ . So without loss of generality, it suffices to assume  $\alpha = \text{id}$  and each  $\Phi_i \in \text{EA}_2^\tau(A^{[m]})$ . We thus write, for each  $1 \leq i \leq q$ , (assuming  $\Phi_i$  is elementary in  $z_1$ )  $\Phi_i = (z_1 + x^{-t_1}P_i(x^{t_2}z_2), z_2)$  for some  $P_i \in A^{[m]}[x^{t_2}z_2] \subset A_\tau[\hat{z}_1]$ .

By assumption 2, for  $i > 1$  we can write  $\omega_i = (x^{-t_1}F_i, x^{-t_2}G_i)$  for some  $F_i, G_i \in R^{[m+2]} \setminus xR^{[m+2]}$ . Given  $Q \in R^{[m]}[z_1, z_2]$ , we will use  $\bar{Q}$  to denote its image (under the quotient map modulo  $x$ ) in  $A^{[m]}[z_1, z_2]$ . Thus, we can interpret assumption 2 as  $\bar{F}_i \neq 0$  and  $\bar{G}_i \neq 0$  for  $1 < i \leq q$ . We inductively show the following claim:

**Claim 26.** For each  $i > 1$ , there exist  $\Phi'_i, \dots, \Phi'_q \in \text{EA}_2^\tau(A^{[m]})$  and  $\rho \in \text{GP}_2^\tau(R^{[m]})$  such that

1.  $\Phi_i \circ \dots \circ \Phi_q = \rho \circ \Phi'_i \circ \dots \circ \Phi'_q$
2. Letting  $\omega'_i = \Phi'_i \circ \dots \circ \Phi'_q \circ \omega$  and setting  $\omega'_i(x^{t_1}z_1) = F'_i$  and  $\omega'_i(x^{t_2}z_2) = G'_i$ , then
  - (a) If  $\Phi'_i$  is nonlinear and elementary in  $z_1$ , then  $\deg \bar{F}'_i > \deg \bar{G}'_i$
  - (b) If  $\Phi'_i$  is nonlinear and elementary in  $z_2$ , then  $\deg \bar{F}'_i < \deg \bar{G}'_i$

Let us first see how this claim completes the lemma. Applying the claim for  $i = 2$ , we obtain

$$\Phi_1 \circ \dots \circ \Phi_q = \Phi_1 \circ \rho \circ \Phi'_2 \circ \dots \circ \Phi'_q = \rho \circ \Phi'_1 \circ \Phi'_2 \circ \dots \circ \Phi'_q$$

with the final equality coming from Corollary 18. Here, each  $\Phi'_i \in \text{EA}_2^\tau(A^{[m]})$ .

Note that it suffices to assume  $\rho = \text{id}$ . Without loss of generality, assume  $\Phi'_1$  is elementary in  $z_1$  and  $\Phi'_2$  is elementary in  $z_2$ . Then we compute

$$\omega_1(x^{t_1}z_1) = (\Phi'_1 \circ \omega'_2)(x^{t_1}z_1) = \omega'_2(x^{t_1}z_1 + P_1(x^{t_2}z_2)) = F'_2 + P_1(G'_2)$$

But assumption 3 implies that  $\bar{F}'_2 + P_1(\bar{G}'_2) \equiv 0$ , and thus  $\deg \bar{F}'_2 = (\deg P_1)(\deg \bar{G}'_2)$ . Since the claim yields that if  $\Phi'_2$  is nonlinear, then  $\deg \bar{F}'_2 < \deg \bar{G}'_2$ , we must have that  $\Phi'_2$  is linear. Let  $b \geq 2$  be minimal such that  $\Phi'_{b+1}$  is non-linear (and thus  $\Phi_2, \dots, \Phi_b$  are all linear). We will derive a contradiction, showing no such  $b$  exists, in which case  $\Phi'_2, \dots, \Phi'_q$  are all linear.

Set  $\beta = \Phi'_2 \circ \dots \circ \Phi'_b = (\beta_{11}z_1 + \beta_{12}x^{t_2-t_1}z_2, \beta_{22}z_2 + \beta_{21}x^{t_1-t_2}z_1) \in \text{GL}_2^\tau(A^{[m]})$  (for some  $\beta_{ij} \in A^{[m]}$ ). Note that by Lemma 22, if  $\beta_{21} = 0$  or  $\beta_{22} = 0$ , we may (absorbing a resulting permutation into  $\rho$ ) replace  $\beta$  by the identity. Thus, we assume without loss of generality that  $\beta_{21} \neq 0$  and  $\beta_{22} \neq 0$ .

But then we have

$$\begin{aligned} F'_2 &= \omega'_2(x^{t_1}z_1) = (\beta \circ \omega'_{b+1})(x^{t_1}z_1) = \omega'_{b+1}(\beta_{11}x^{t_1}z_1 + \beta_{12}x^{t_2}z_2) = \beta_{11}F'_{b+1} + \beta_{12}G'_{b+1} \\ G'_2 &= \omega'_2(x^{t_2}z_2) = (\beta \circ \omega'_{b+1})(x^{t_2}z_2) = \omega'_{b+1}(\beta_{22}x^{t_2}z_2 + \beta_{21}x^{t_1}z_1) = \beta_{21}F'_{b+1} + \beta_{22}G'_{b+1} \end{aligned} \quad (6)$$

Thus, since  $\overline{F'_2} + P_1(\overline{G'_2}) = 0$ , we have from (6)

$$\beta_{11}\overline{F'_{b+1}} + \beta_{12}\overline{G'_{b+1}} + P_1(\beta_{21}\overline{F'_{b+1}} + \beta_{22}\overline{G'_{b+1}}) = 0 \quad (7)$$

Since  $\Phi'_{b+1}$  is nonlinear, we must have (by the claim)  $\deg \overline{F'_{b+1}} \neq \deg \overline{G'_{b+1}}$ ; then since  $\beta_{21} \neq 0$  and  $\beta_{22} \neq 0$ , from (7) we see

$$\max \left\{ \deg \overline{F'_{b+1}}, \deg \overline{G'_{b+1}} \right\} \geq \deg \left( \beta_{11}\overline{F'_{b+1}} + \beta_{12}\overline{G'_{b+1}} \right) = (\deg P_1) \max \left\{ \deg \overline{F'_{b+1}}, \deg \overline{G'_{b+1}} \right\}$$

Thus we must have  $\deg P_1 = 1$ . Let  $P_1(z) = \mu z$  for some  $\mu \in A^{[m]}$ . Then (again since  $\deg \overline{F'_{b+1}} \neq \deg \overline{G'_{b+1}}$ ) we see  $\beta_{11} + \mu\beta_{21} = 0$  and  $\beta_{12} + \mu\beta_{22} = 0$ ; however, this implies  $\det \beta = \det \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = 0$ , contradicting  $\beta \in \text{GL}_2^\tau(R^{[m]})$ .

So we now have that  $\Phi'_2, \dots, \Phi'_q$  are all linear. We will continue to write  $\beta = \Phi'_2 \circ \dots \circ \Phi'_q \in \overline{\text{GL}_2^\tau}(R^{[m]})$ . Now write  $\omega(x^{t_{i,1}}z_1) = F$  and  $\omega(x^{t_{i,2}}z_2) = G$ . We have  $F, G \in R^{[m+2]}$ , but by assumption 1, either  $\overline{F} = 0$  or  $\overline{G} = 0$ . Then we compute

$$\omega_1(x^{t_1}z_1) = (\Phi_1 \circ \beta \circ \omega)(x^{t_1}z_1) = \omega(\beta_{11}x^{t_1}z_1 + \beta_{12}x^{t_2}z_2 + P_1(\beta_{21}x^{t_1}z_2 + \beta_{22}x^{t_2}z_2)) = \beta_{11}F + \beta_{12}G + P_1(\beta_{21}F + \beta_{22}G)$$

But since  $\overline{F} = 0$  or  $\overline{G} = 0$ , we clearly must have  $P_1$  is linear. We may then write  $\Phi'_1 \circ \beta = (az_1 + bx^{t_2-t_1}z_2, cx^{t_1-t_2}z_1 + dz_2)$  (for some  $a, b, c, d \in A^{[m]}$ ), and compute  $\omega_1(x^{t_1}z_1) = aF + bG$ . Since assumption 3 implies  $a\overline{F} + b\overline{G} = 0$ , and either  $\overline{F} = 0$  or  $\overline{G} = 0$  (but not both), we must have  $a = 0$  or  $b = 0$ . Then from Lemma 22, we have

$$\Phi'_1 \circ \dots \circ \Phi'_q = \beta = \rho \circ \Phi$$

for some  $\rho \in \text{GL}_2^\tau(R^{[m]})$  and elementary  $\Phi \in \text{EA}_2^\tau(R^{[m]}) \cap \text{GL}_2^\tau(R^{[m]})$ , as required.

We thus are reduced to proving Claim 26. □

*Proof of Claim 26.* The proof is induction on  $i$ . First, suppose  $i = q$ . We set  $\Phi'_q = \Phi_q$ , and without loss of generality, assume  $\Phi'_q$  is elementary in  $z_1$  (the case where it is elementary in  $z_2$  follows similarly). Write  $\omega = (x^{-t_1}F, x^{-t_2}G)$  for some  $F, G \in R^{[m+2]}$ . Note that our assumption that  $\Phi_q$  is elementary in  $z_1$  (along with assumption 2) forces  $\overline{F} = 0$  and  $\overline{G} \neq 0$ . Then

$$\omega'_q = \Phi'_q \circ \omega = (x^{-t_1}(F + P_q(G)), x^{-t_2}G)$$

Since  $F \in (x)$ , we thus have  $\overline{F'_q} = P_q(\overline{G})$  and  $\overline{G'_q} = \overline{G}$ . Then if  $\Phi'_q$  is non-linear,  $\deg P_q > 1$ , and we have  $\deg \overline{F'_q} = (\deg P_q)(\deg \overline{G}) > \deg \overline{G'_q}$  as required.

Now suppose  $i < q$  with  $\Phi_i$  non-linear. By the induction hypothesis, we may replace  $\Phi_j$  with  $\Phi'_j$  for  $j > i$  with the desired properties (using Corollary 18 to push any resulting permutation to the left). Let  $j > i$  be minimal such that  $\Phi'_j$  is also non-linear. Let  $\beta = \Phi'_{i+1} \circ \dots \circ \Phi'_{j-1} = (\beta_{11}z_1 + \beta_{12}x^{t_1-t_2}z_2, \beta_{22}z_2 + \beta_{21}x^{t_1-t_2}z_1) \in \text{GL}_2^\tau(A^{[m]})$  for some  $\beta_{i,j} \in A^{[m]}$ . Without loss of generality, assume  $\Phi_i$  is elementary in  $z_1$ . Then by Lemma 22, we may assume  $\beta_{21} \neq 0$  and  $\beta_{22} \neq 0$  by factoring through a permutation. We then compute

$$\begin{aligned} \omega_i &= \Phi_i \circ \beta \circ \omega'_j \\ &= \Phi_i \circ (x^{-t_1}(\beta_{11}F'_j + \beta_{12}G'_j), x^{-t_2}(\beta_{21}F'_j + \beta_{22}G'_j)) \\ &= (x^{-t_1}(\beta_{11}F'_j + \beta_{12}G'_j + P_i(\beta_{21}F_j + \beta_{22}G_j)), x^{-t_2}(\beta_{21}F'_j + \beta_{22}G'_j)) \end{aligned}$$

Then clearly we have

$$\begin{aligned} F_i &= \beta_{11}F'_j + \beta_{12}G'_j + P_i(\beta_{21}F_j + \beta_{22}G_j) \\ G_i &= \beta_{21}F'_j + \beta_{22}G'_j \end{aligned}$$

Let  $d = \max\{\deg \overline{F'_j}, \deg \overline{G'_j}\}$ . Then since  $\deg \overline{G'_j} \neq \deg \overline{F'_j}$  by the inductive hypothesis, and since  $\beta_{21} \neq 0$  and  $\beta_{22} \neq 0$ ,  $\deg \overline{G_i} = d$  and  $\deg \overline{F_i} = (\deg P_i)d > d$  since  $P_i$  is non-linear. Thus we may now take  $\Phi'_i = \Phi_i$  to complete the proof.  $\square$

### 3.2 Main Theorems

We can now state and prove our main theorems.

**Main Theorem 1.** *Let  $\tau_0 \geq \dots \geq \tau_q \in \mathbb{N}^n$ . For  $0 \leq i \leq q$ , let  $\Phi_i \in \text{GA}_n^{\tau_i}(R^{[m]})$  and  $\alpha_i \in \text{IA}_{m+n}^{\tau_i}(R)$ . Set*

$$\psi = \alpha_0 \circ \Phi_0 \circ \dots \circ \alpha_q \circ \Phi_q$$

*Then  $(\psi(y_1), \dots, \psi(y_m))$  is a partial system of coordinates over  $R$ . Moreover, if  $A$  is a regular domain, and  $\alpha_i, \Phi_i \in \text{TA}_{m+n}(S)$  for  $0 \leq i \leq q$ , then  $(\psi(y_1), \dots, \psi(y_m))$  can be extended to a stably tame automorphism of  $R^{[m+n]}$ .*

We are now ready to prove the following result, which immediately implies Main Theorem 1. The inclusion of the permutation maps  $\rho_i$  is not necessary to achieve Main Theorem 1, but will help us in our proof of Main Theorem 2. Note that if we assume each  $\rho_i$  is of the form in Definition 4, then we may drop the assumption “ $A$  is a connected, reduced ring”. In particular, we do not need to assume  $A$  is connected and reduced in Main Theorem 1, since we set  $\rho_i = \text{id}$  for each  $i$  to obtain it from Theorem 27.

**Theorem 27.** *Let  $A$  be a connected, reduced ring, and let  $\tau_0, \dots, \tau_q \in \mathbb{N}^n$ . Let  $\rho_i \in \text{GP}_n(S^{[m]})$ ,  $\alpha_i \in \text{IA}_{m+n}^{\tau_i}(R)$ , and  $\Phi_i \in \text{GA}_n^{\rho_i(\tau_i)}(R^{[m]})$  for each  $0 \leq i \leq q$ . Set*

$$\psi_i = \alpha_0 \circ \rho_0 \circ \Phi_0 \circ \dots \circ \alpha_i \circ \rho_i \circ \Phi_i$$

*Suppose  $\rho_i(\tau_i) \geq \tau_{i+1}$  for each  $0 \leq i \leq q$ . Then for each  $0 \leq i \leq q$ , there exists  $\theta_i \in \text{IA}_{m+n}^{\tau_{i+1}}(R) \cap \text{IA}_{m+n}(R)$  with  $\theta_i(y_j) = \psi_i(y_j)$  for each  $1 \leq j \leq m$ . Moreover, if  $\alpha_k, \Phi_k \in \text{TA}_{m+n}(S)$  for  $0 \leq k \leq i$ , then  $\theta_i$  is stably tame.*

*Proof.* The proof is by induction on  $i$ . Note that we may use a trivial base case of  $i = -1$  and  $\theta_{-1} = \text{id}$ . So we suppose  $i \geq 0$ . By the induction hypothesis we have  $\theta_{i-1} \in \text{IA}_{m+n}^{\tau_i}(R)$ . Thus,  $(\theta_{i-1} \circ \alpha_i) \in \text{IA}_{m+n}^{\tau_i}(R)$ , and by Corollary 16,  $\rho_i^{-1} \circ (\theta_{i-1} \circ \alpha_i) \circ \rho_i \in \text{IA}_{m+n}^{\rho_i(\tau_i)}(R)$ . Since  $\Phi_i \in \text{GA}_n^{\rho_i(\tau_i)}(R)$  and  $\tau_{i+1} \leq \rho(\tau_i)$ , we can apply Theorem 13 to obtain  $\tilde{\Phi} \in \text{GA}_n^{\rho_i(\tau_i)}(R^{[m]})$  such that

$$\theta_i := \tilde{\Phi} \circ (\rho_i^{-1} \circ \theta_{i-1} \circ \alpha_i \circ \rho_i) \circ \Phi_i \in \text{IA}_{m+n}^{\tau_{i+1}}(R) \cap \text{IA}_{m+n}(R)$$

Noting that  $\tilde{\Phi}, \rho_i \in \text{GA}_n(S^{[m]})$  and thus fix each  $y_j$ , and by the inductive hypothesis  $\theta_{i-1}(y_j) = \psi_{i-1}(y_j)$  we have

$$\begin{aligned} \theta_i(y_j) &= (\tilde{\Phi} \circ \rho_i^{-1} \circ \theta_{i-1} \circ \alpha_i \circ \rho_i \circ \Phi_i)(y_j) \\ &= (\theta_{i-1} \circ \alpha_i \circ \rho_i \circ \Phi_i)(y_j) \\ &= (\psi_{i-1} \circ \alpha_i \circ \rho_i \circ \Phi_i)(y_j) \\ &= \psi_i(y_j) \end{aligned}$$

for each  $1 \leq j \leq m$ . Moreover, if  $\alpha_0, \Phi_0, \dots, \alpha_i, \Phi_i \in \text{TA}_{m+n}(S)$ , then the inductive hypothesis along with Theorem 13 guarantee  $\theta_i \in \text{TA}_{m+n}(S)$  as well. Noting that since  $\theta_i \in \text{IA}_{m+n}(R)$  we have  $\theta_i \equiv \text{id} \pmod{x}$ , the stable tameness assertion follows immediately from the following result of Berson, van den Essen, and Wright:

**Theorem 28** ([3], Theorem 4.5). *Let  $A$  be a regular domain, and let  $\phi \in \text{GA}_n(R)$  with  $J\phi = 1$ . If  $\phi \in \text{TA}_n(S)$  and  $\tilde{\phi} \in \text{EA}_n(R/xR)$ , then  $\phi$  is stably tame.*

□

**Main Theorem 2.** Suppose  $A$  is an integral domain of characteristic zero. Let  $\Phi_0, \dots, \Phi_q \in \text{EA}_2(S^{[m]})$  be elementaries. Let  $\alpha_i \in \text{GA}_{m+2}(S)$  and  $\rho_i \in \text{GP}_2(S^{[m]})$  for each  $0 \leq i \leq q$ . Set

$$\omega_i = \alpha_i \circ \rho_i \circ \Phi_i \circ \dots \circ \alpha_q \circ \rho_q \circ \Phi_q$$

and define  $\tau_i \in \mathbb{N}^2$  to be minimal such that  $\omega_i(A_{\tau_i}) \subset R^{[m+2]}$  for  $0 \leq i \leq q$ . If  $\alpha_i \in \text{IA}_{m+2}^{\tau_i}(R)$  for each  $0 \leq i \leq q$ , then there exists  $\theta \in \text{IA}_{m+2}(R)$  such that  $\theta(y_j) = \omega_0(y_j)$ .

The theorem follows from following claim, which allows us to apply Theorem 27 to  $\omega_0$ . By convention, we will let  $\tau_{q+1} = 0$ .

**Claim 29.** For each  $a \leq q$ , there exist the following:

1. A sequence  $\tilde{\tau}_a, \dots, \tilde{\tau}_q \in \mathbb{N}^2$
2.  $\tilde{\rho}_a, \dots, \tilde{\rho}_q \in \text{GP}_2(S^{[m]})$
3. For each  $a \leq i \leq q$ ,  $\tilde{\alpha}_i \in \text{IA}_{m+2}^{\tilde{\tau}_i}(R)$  and  $\Phi_i \in \text{EA}_2^{\tilde{\rho}_i(\tilde{\tau}_i)}(R^{[m]})$

such that

1.  $\tilde{\rho}_i(\tilde{\tau}_i) \geq \tilde{\tau}_{i+1}$
2. Setting  $\tilde{\omega}_i = \tilde{\alpha}_i \circ \tilde{\rho}_i \circ \tilde{\Phi}_i \circ \dots \circ \tilde{\alpha}_q \circ \tilde{\rho}_q \circ \tilde{\Phi}_q$ , each  $\tilde{\tau}_i$  is minimal such that  $\tilde{\omega}_i(A_{\tilde{\tau}_i}) \subset R^{[m+2]}$ , for  $a \leq i \leq q$
3.  $\omega_a = \tilde{\omega}_a$

*Proof of Claim 29.* First, suppose  $\rho(\tau_i) \geq \tau_{i+1}$  for  $a \leq i \leq q$ . Then all we need to show is that  $\Phi_i \in \text{EA}_2^{\rho_i(\tau_i)}(R^{[m]})$ . Note that by Corollary 17 it is equivalent to show that  $\Phi'_i := \rho_i \circ \Phi_i \circ \rho_i^{-1} \in \text{EA}_2^{\tau_i}(R)$ .

Without loss of generality, write  $\Phi'_i = (z_1 + x^{-s}P(x^{t_{i,2}}z_2), z_2)$  for some  $P(x^{t_{i,2}}z_2) \in A_{\tau_i}[\hat{z}_1] \setminus xA_{\tau_i}[\hat{z}_1]$ . Then

$$\begin{aligned} (\rho_i \circ \Phi_i \circ \omega_{i+1})(x^{t_{i,1}}z_1) &= (\Phi'_i \circ \rho_i \circ \omega_{i+1})(x^{t_{i,1}}z_1) \\ &= (\rho_i \circ \omega_{i+1})(x^{t_{i,1}}z_1) + x^{t_{i,1}-s}P((\rho_i \circ \omega_{i+1})(x^{t_{i,2}}z_2)) \end{aligned}$$

Since  $\rho_i(\tau_i) \geq \tau_{i+1}$ , we have  $\rho_i(A_{\tau_i}) = A_{\rho_i(\tau_i)} \subset A_{\tau_{i+1}}$ . In particular,  $(\rho_i \circ \omega_{i+1})(A_{\tau_i}) = \omega_{i+1}(A_{\rho_i(\tau_i)}) \subset \omega_{i+1}(A_{\tau_{i+1}}) \subset R^{[m+2]}$ . As  $(\rho_i \circ \Phi_i \circ \omega_{i+1})(A_{\tau_i}) \subset R^{[m+2]}$ , this implies that  $x^{t_{i,1}-s}P((\rho_i \circ \omega_{i+1})(x^{t_{i,2}}z_2)) \in R^{[m+2]}$  as well. Thus  $t_{i,1} \geq s$  since  $P \notin (x)$  and  $(\rho_i \circ \omega_{i+1})(x^{t_{i,2}}z_2) \in R^{[m+2]} \setminus xR^{[m+2]}$  (by Lemma 23); but  $t_{i,1} \geq s$  is precisely the condition that  $\Phi_i \in \text{EA}_2^{\tau_i}(R^{[m]})$  as required.

It now suffices to assume that  $a \leq q$  is maximal such that  $\rho_a(\tau_a) < \tau_{a+1}$ ; then  $\rho_i(\tau_i) \geq \tau_{i+1}$  and  $\Phi_i \in \text{EA}_2^{\rho_i(\tau_i)}(R^{[m]})$  (by the above argument) for  $a+1 \leq i \leq q$ .

We proceed by induction downwards on  $q - a$ . Let  $b > a$  be minimal such that  $\rho_b(\tau_b) > \tau_{b+1}$ . That is,  $\rho_i(\tau_i) = \tau_{i+1}$  for  $a < i < b$ . We will show that we can replace  $\alpha_a \circ \rho_a \circ \Phi_a \circ \dots \circ \alpha_b \circ \rho_b \circ \Phi_b$  by a single  $\alpha'_a \circ \rho'_a \circ \Phi'_a$ ; then the induction hypothesis will imply that  $\omega_a$  is in the desired form. Note that by Lemma 21 (with  $\sigma = \rho_a(\tau_a)$ ,  $\tau = \tau_{a+1}$ , and  $\omega = \omega_{a+1}$ ), we must have  $\Phi_a \in \text{EA}_2^{\tau_{a+1}}(R^{[m]})$ . Also, since  $\rho_i(\tau_i) = \tau_{i+1}$  for  $a < i < b$ ,  $\Phi_i \in \text{EA}_2^{\tau_{i+1}}(R^{[m]})$  for  $a < i < b$ . Then by Lemma 19, it suffices to assume that  $\alpha_{a+1} = \dots = \alpha_b = \text{id}$ ,  $\rho_{a+1} = \dots = \rho_b = \text{id}$ ,  $\rho_a(\tau_a) < \tau_{a+1} = \dots = \tau_b > \tau_{b+1}$  and  $\Phi_i \in \text{EA}_n^{\tau_i}(R^{[m]})$  for  $a \leq i \leq b$ .

A priori, it seems we may no longer be able to assume the minimality condition on the  $\tau_i$  when  $a < i < b$ . However, we may simply replace the  $\tau_i$  by the minimal  $\tau_i$  such that  $\omega_i(A_{\tau_i}) \subset R^{[m+2]}$  (for  $a \leq i \leq b$ ). Then we may need to increase  $a$  (but it will not exceed  $b$ ) such that we may still assume  $\alpha_{a+1} = \dots = \alpha_b = \text{id}$ ,  $\rho_{a+1} = \dots = \rho_b = \text{id}$ ,  $\Phi_a, \dots, \Phi_b \in \text{EA}_n^{\tau_b}(R^{[m]})$ , and

$$\rho_a(\tau_a) < \tau_{a+1} = \dots = \tau_b > \tau_{b+1}$$

We also now see that

$$\omega_a = \alpha_a \circ \rho_a \circ \Phi_a \circ \Phi_{a+1} \circ \dots \circ \Phi_b \circ \omega_{b+1} \tag{8}$$

Set

$$\tau_b = (t_1, t_2) \tag{9}$$

for some  $t_1, t_2 \in \mathbb{N}$ . Without loss of generality, assume  $\Phi_a$  is elementary in  $z_1$ . Then since  $\rho_a(\tau_a) < \tau_{a+1} = \tau_b = (t_1, t_2)$ , the minimality of  $\tau_a$  implies  $(\Phi_a \circ \cdots \circ \Phi_b \circ \omega_{b+1})(x^{t_1} z_1) \in xR^{[m+2]}$ . Then, by Lemma 25, we may assume that  $\Phi_a \circ \cdots \circ \Phi_b = \alpha \circ \rho \circ \Phi$  for some  $\alpha \in \text{IA}_2^{\tau_{a+1}}(R^{[m]})$ ,  $\rho \in \text{GP}_2^{\tau_{a+1}}(R^{[m]})$  and elementary  $\Phi \in \text{EA}_2^{\tau_{a+1}}(R^{[m]}) \cap \text{GL}_2^{\tau_{a+1}}(R^{[m]})$ . Then

$$\omega_a = \alpha_a \circ \rho_a \circ \alpha \circ \rho \circ \Phi \circ \omega_{b+1}$$

Noting that  $\rho_a(\tau_a) < \tau_{a+1}$ , by Corollary 20, we have  $\alpha_a \circ \rho_a \circ \alpha \circ \rho = \alpha'_a \circ \rho'_a \circ \Phi'$  for some  $\alpha'_a \in \text{IA}_{m+2}^{\tau_a}(R)$ ,  $\rho'_a = \rho_a \circ \rho \in \text{GP}_2(S^{[m]})$ , and  $\Phi' \in \text{EA}_2^{\tau_{a+1}}(R^{[m]})$  (since  $\rho(\tau_{a+1}) = \tau_{a+1}$ ). Thus we have

$$\omega_a = \alpha'_a \circ \rho'_a \circ \Phi' \circ \Phi \circ \omega_{b+1}$$

First, suppose  $\Phi'$  and  $\Phi$  are both elementary in the same variable; then we may set  $\tilde{\Phi} = \Phi' \circ \Phi$  and  $\tilde{\Phi} \in \text{EA}_2^{\tau_{a+1}}(R^{[m]})$  is elementary, and

$$\omega_a = \alpha'_a \circ \rho'_a \circ \tilde{\Phi} \circ \omega_{b+1} \tag{10}$$

Similarly, if we suppose instead that  $\Phi'$  and  $\Phi$  are elementary in different variables, then since  $\Phi \in \text{GL}_2^{\tau_{a+1}}(R^{[m]})$ , by Lemma 22 there exist  $\tilde{\rho} \in \text{GP}_2^{\tau_{a+1}}(R^{[m]})$  and  $\tilde{\Phi} \in \text{EA}_2^{\tau_{a+1}}(R^{[m]})$  such that  $\Phi' \circ \Phi = \tilde{\rho} \circ \tilde{\Phi}$ . Then we have

$$\omega_a = \alpha'_a \circ (\rho'_a \circ \tilde{\rho}) \circ \tilde{\Phi} \circ \omega_{b+1} \tag{11}$$

Note that since  $\tilde{\rho} \in \text{GP}_2^{\tau_{a+1}}(R^{[m]})$  that  $\rho'_a(\tau_a) < \tau_{a+1}$  implies  $(\rho'_a \circ \tilde{\rho})(\tau_a) < \tau_{a+1}$ . Thus, in either case, we see we have written  $\omega_a$ , which was originally (8) a product of  $q - a + 1$  elementaries, in (10) or (11) as a product with  $q - b$  elementaries, and since  $a < b$ , we must have  $q - a + 1 > q - b$ . The induction hypothesis then completes the proof.  $\square$

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